```
# Write a recursive function that takes as input a
# non-negative number (num) and returns the number of
# occurrences of the digit 1. countOnes(123114) = 3

def countOnes(num):
```

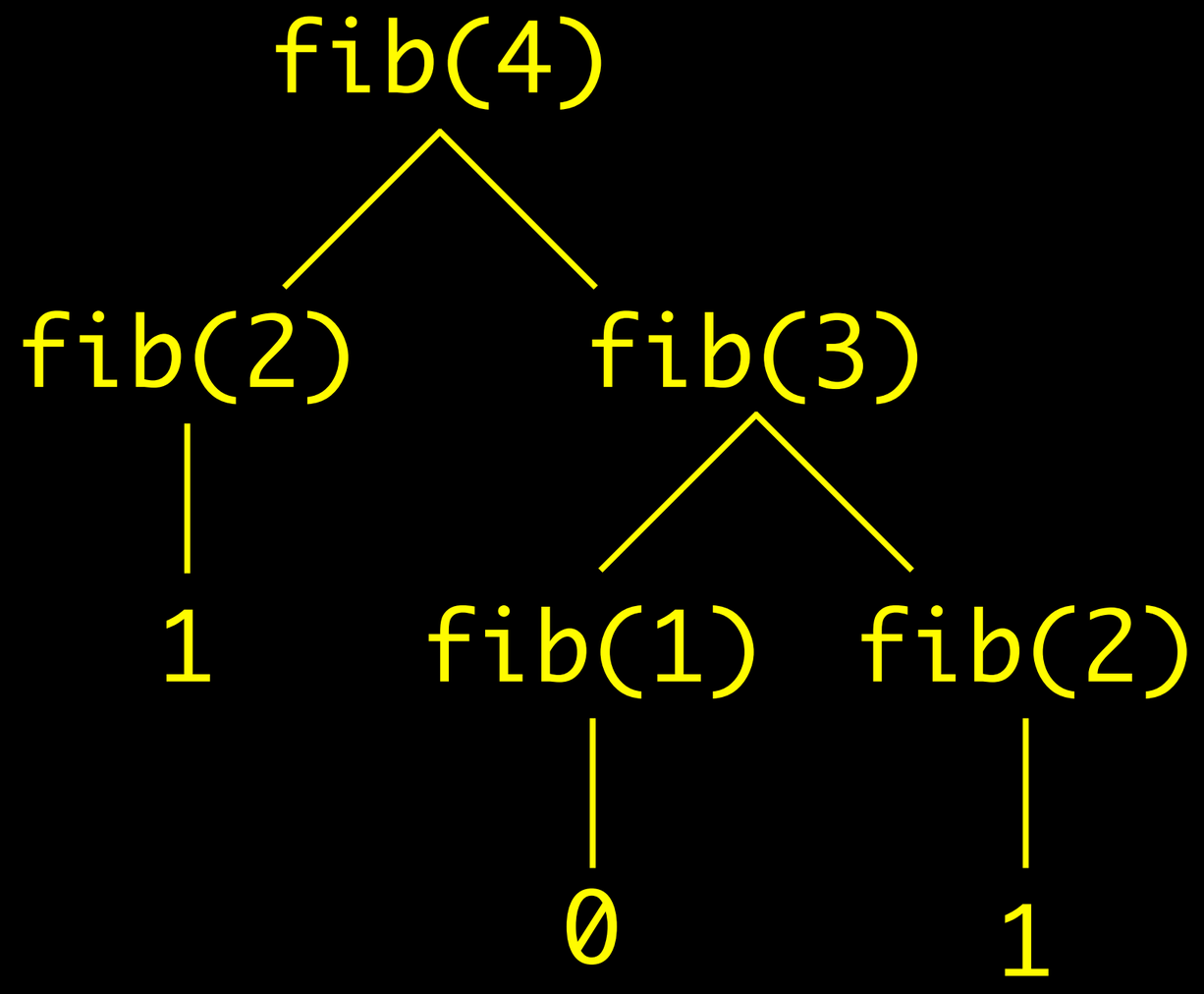
```
# Write a recursive function that takes as input a
# non-negative number (num) and returns the number of
# occurrences of the digit 1. countOnes(123114) = 3

def countOnes(num):
    if num == 0:
        return 0
    else:
```

```
# Write a recursive function that takes as input a
# non-negative number (num) and returns the number of
# occurrences of the digit 1. countOnes(123114) = 3

def countOnes(num):
    if num == 0:
        return 0
    else:
        if num % 10 == 1
            return 1 + countOnes(num//10)
```

```
# Write a recursive function that takes as input a
# non-negative number (num) and returns the number of
# occurrences of the digit 1. countOnes(123114) = 3

def countOnes(num):
    if num == 0:
        return 0
    else:
        if num % 10 == 1
            return 1 + countOnes(num//10)
        else:
            return countOnes(num//10)
```

Tree recursion: Fibonacci sequence

```
def fib(n):
    if n == 1:
        return 0
    elif n == 2:
        return 1
    else:
        return fib(n-2) + fib(n-1)
```



Tree recursion: count partitions

The number of partitions of a positive integer n, using parts up to size m, is the number of ways in which n can be expressed as the sum of positive integer parts up to m in non-decreasing order.

```
cp(4,2)
    1 + 1 + 1 + 1
    1 + 1 + 2
    2 + 2
```

Tree recursion: count partitions

```
cp(6,4)
    1 + 1 + 1 + 1 + 1 + 1
    1 + 1 + 1 + 1 + 2
    1 + 1 + 2 + 2
    2 + 2 + 2
    1 + 1 + 1 + 3
    1 + 2 + 3
    3 + 3
    1 + 1 + 4
    2 + 4
```

Tree recursion: count partitions

```
cp(6,4)
    1 + 1 + 1 + 1 + 1 + 1   # don't use 4
    1 + 1 + 1 + 1 + 2
    1 + 1 + 2 + 2
    2 + 2 + 2
    1 + 1 + 1 + 3
    1 + 2 + 3
    3 + 3
    1 + 1 + 4
    2 + 4                   # use 4
```

Tree recursion: count partitions

```
cp(6,4)
    1 + 1 + 1 + 1 + 1 + 1    # don't use 4: cp(6,3)
    1 + 1 + 1 + 1 + 2
    1 + 1 + 2 + 2
    2 + 2 + 2
    1 + 1 + 1 + 3
    1 + 2 + 3
    3 + 3
    1 + 1 + 4
    2 + 4                    # use 4: cp(6-4,4)
```

Tree recursion: count partitions

```
cp(6,4)
    1 + 1 + 1 + 1 + 1 + 1    # don't use 3: cp(6,2)
    1 + 1 + 1 + 1 + 2
    1 + 1 + 2 + 2
    2 + 2 + 2
    1 + 1 + 1 + 3            # use 3: cp(6-3,3)
    1 + 2 + 3
    3 + 3
```

Tree recursion: partitions

```
def cp(n, m):
    if n == 0:
        return 1
    elif n < 0 or m == 0:
        return 0
    else:
        return + cp(n, m-1) + cp(n-m, m)
```

```
# mutual recursion: Luhn sum (check sum)

7  9  9  2  7  3  9  8  7  1  3# acct number
```

```
# mutual recursion: Luhn sum (check sum)

7  9  9  2  7  3  9  8  7  1  3 # acct number
  18     4     6    16     2    # double every other
```

```
# mutual recursion: Luhn sum (check sum)

7  9  9  2  7  3  9  8  7  1  3 # acct number
  18     4     6    16     2    # double every other
   9     4     6     7     2    # sum digits > 10
```

```
# mutual recursion: Luhn sum (check sum)


7  9  9  2  7  3  9  8  7  1  3 # acct number
  18     4     6    16     2    # double every other
   9     4     6     7     2    # sum digits > 10
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70 # sum
```

```
# mutual recursion: Luhn sum (check sum)

7  9  9  2  7  3  9  8  7  1  3 # acct number
  18     4     6    16     2    # double every other
   9     4     6     7     2    # sum digits > 10
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70 # sum

70 % 10 == 0 # valid Luhn sum is multiple of 10
```

```
# mutual recursion: Luhn sum (check sum)


7  9  9  2  7  3  9  8  7  1  3
  18     4     6    16     2
   9     4     6     7     2
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70

luhn_sum(79927398713)
```

```
# mutual recursion: Luhn sum (check sum)


7  9  9  2  7  3  9  8  7  1  3
  18     4     6    16     2
   9     4     6     7     2
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70


luhn_sum(79927398713)
   luhn_sum2(7992739871) + 3
```

```
# mutual recursion: Luhn sum (check sum)


7  9  9  2  7  3  9  8  7  1  3
  18     4     6    16     2
   9     4     6     7     2
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70


luhn_sum(79927398713)
    luhn_sum2(7992739871) + 3
        luhn_sum(799273987) + sum_dig(2*1)
```

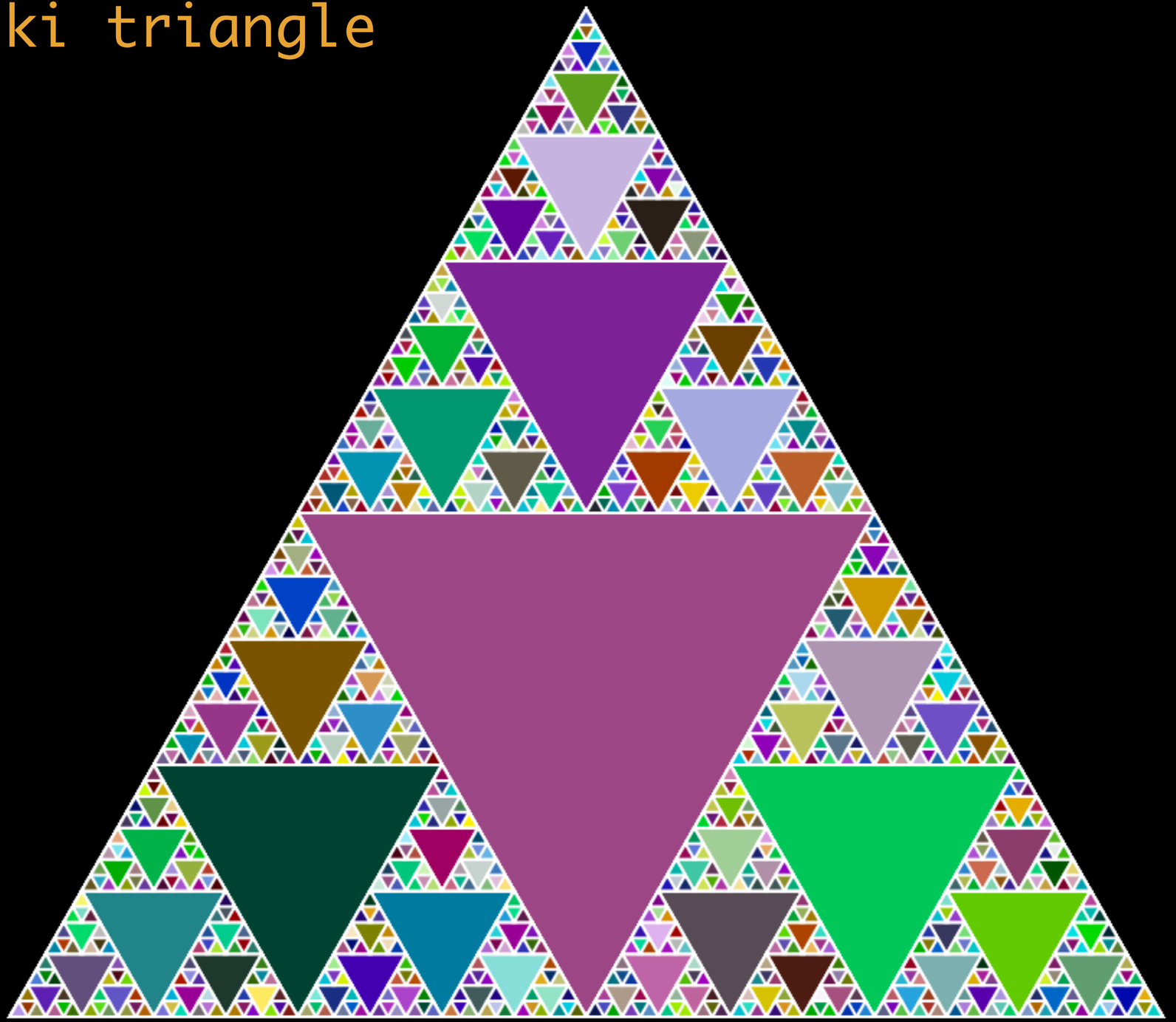
```
# mutual recursion: Luhn sum (check sum)

7  9  9  2  7  3  9  8  7  1  3
  18     4     6    16     2
   9     4     6     7     2
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70

luhn_sum(79927398713)
    luhn_sum2(7992739871) + 3
        luhn_sum(799273987) + sum_dig(2*1)
            luhn_sum2(79927398) + 7
```

```
# mutual recursion: Luhn sum (check sum)

7  9  9  2  7  3  9  8  7  1  3
  18     4     6    16     2
   9     4     6     7     2
7 +9 +9 +4 +7 +6 +9 +7 +7 +2 +3 = 70

luhn_sum(79927398713)
    luhn_sum2(7992739871) + 3
        luhn_sum(799273987) + sum_dig(2*1)
            luhn_sum2(79927398) + 7
                luhn_sum(7992739) + sum_dig(2*8)
```

```
def split(n):
    # Split a positive integer into all but its last digit and
    # its last digit
    # split(123) -> (123 // 10 = 12, 123 % 10 = 3)
    return n // 10, n % 10

def sum_digits(n):
    # Return the sum of the digits of positive integer n
    if n < 10:
        return n
    else:
        a, b = split(n)
        return sum_digits(a) + b
```

```
def luhn_sum(n):
    if n < 10:
        return n
    else:
        a, b = split(n)
        return luhn_sum2(a) + b

def luhn_sum2(n):
    a, b = split(n)
    d = sum_digits(2 * b)
    if n < 10:
        return d
    else:
        return luhn_sum(a) + d
```

Sierpiński triangle

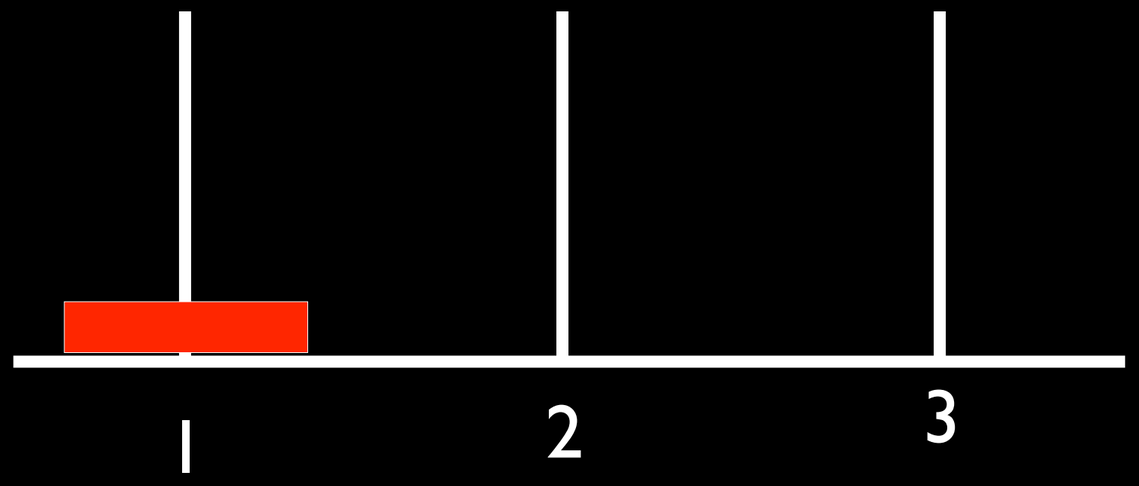


Towers of Hanoi

http://haubergs.com/hanoi

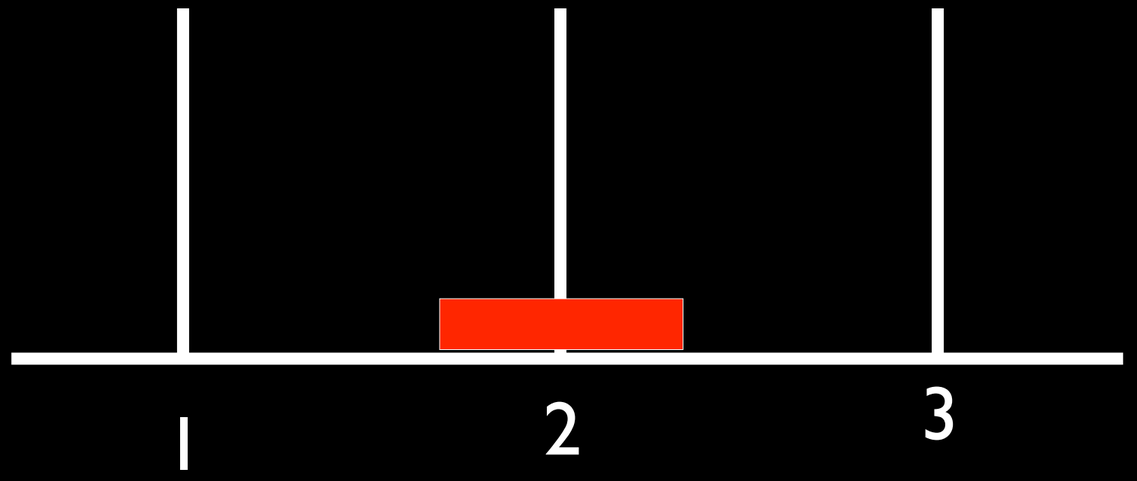
Towers of Hanoi

n = 1: move disk from post 1 to post 2



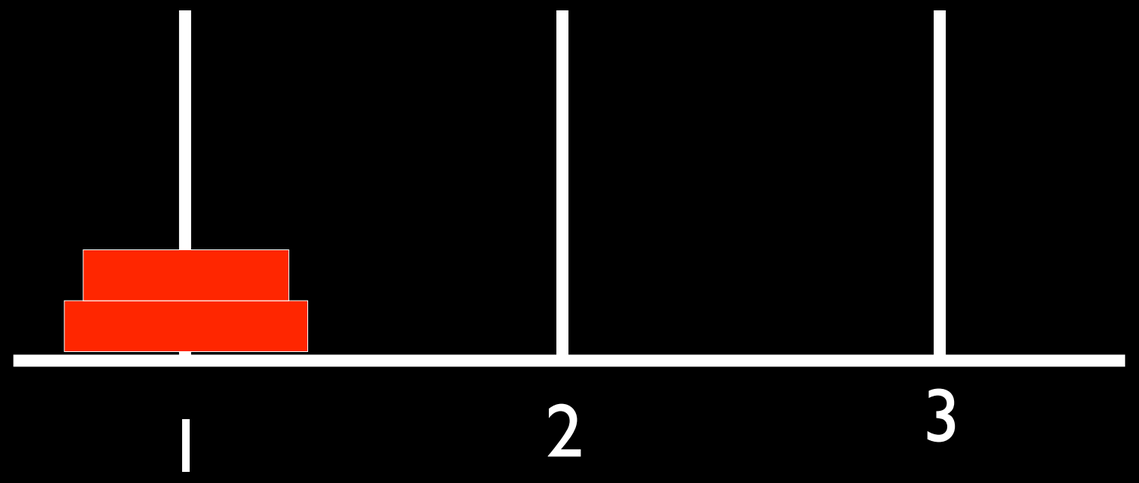
Towers of Hanoi

n = 1: move disk from post 1 to post 2



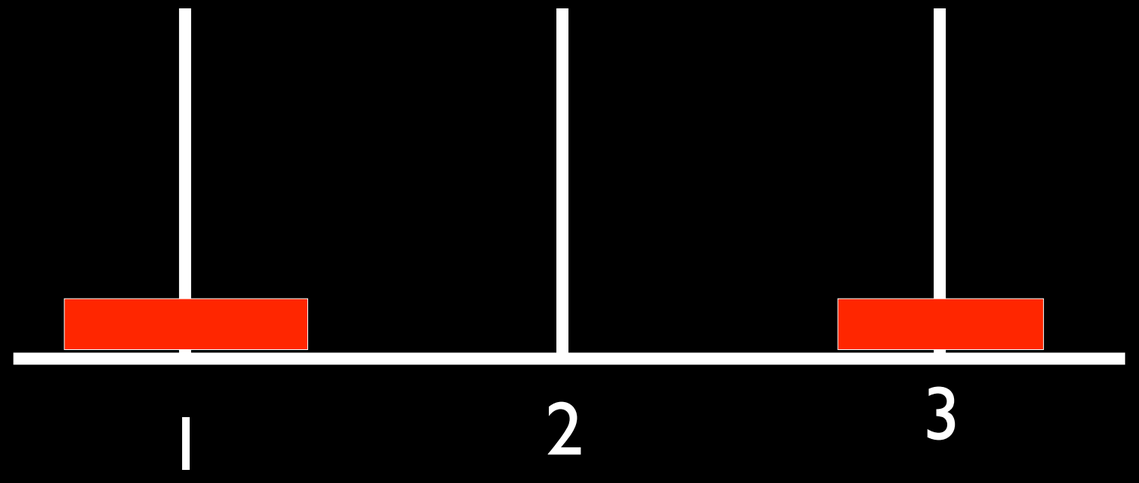
Towers of Hanoi

n = 2: move disks from post 1 to post 2



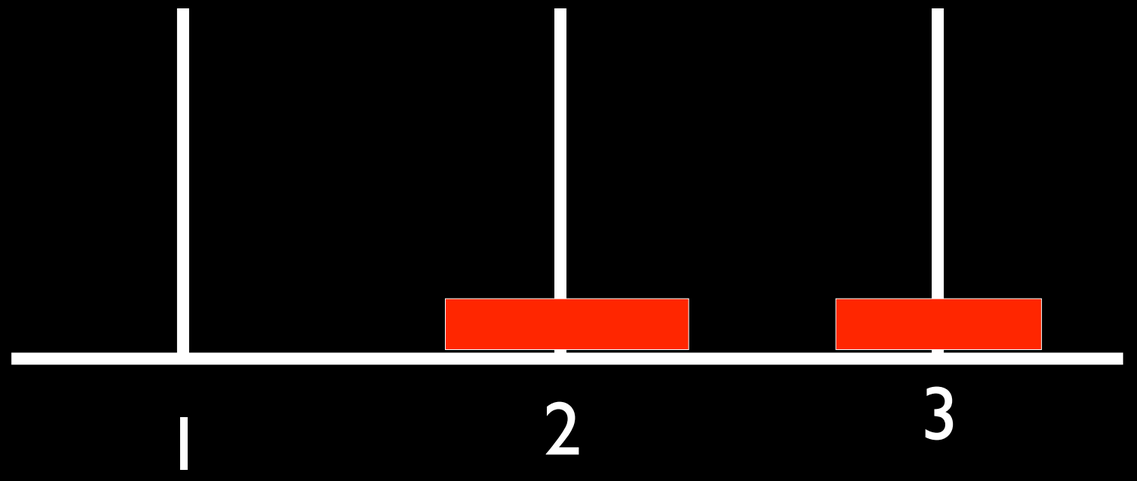
Towers of Hanoi

n = 2: move disks from post 1 to post 2



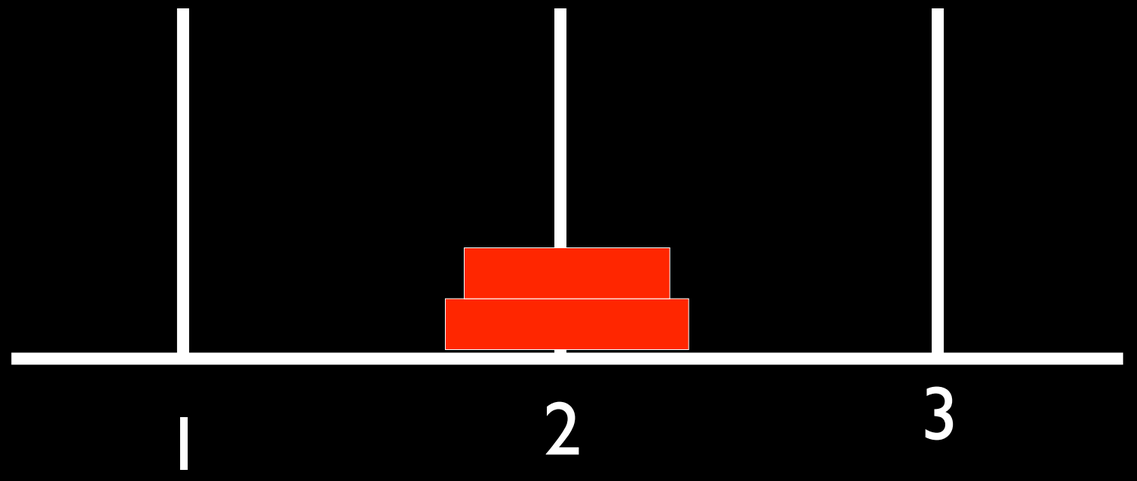
Towers of Hanoi

n = 2: move disks from post 1 to post 2



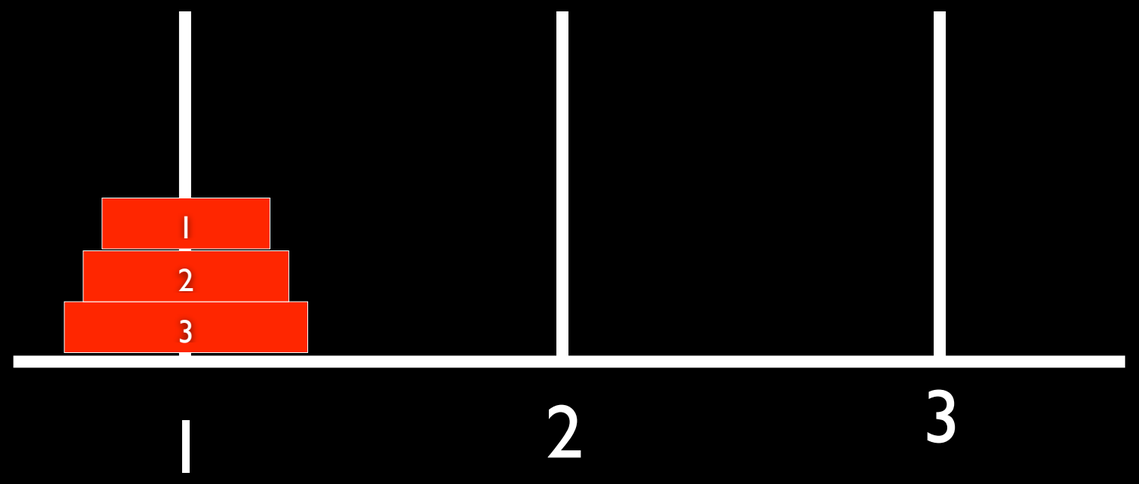
Towers of Hanoi

n = 2: move disks from post 1 to post 2



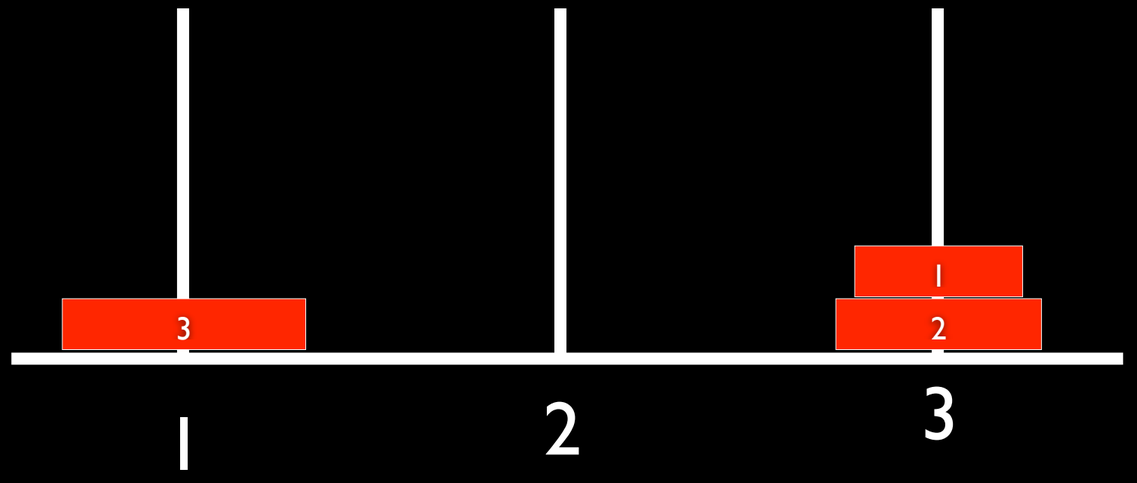
Towers of Hanoi

n = 3: move disks from post 1 to post 2



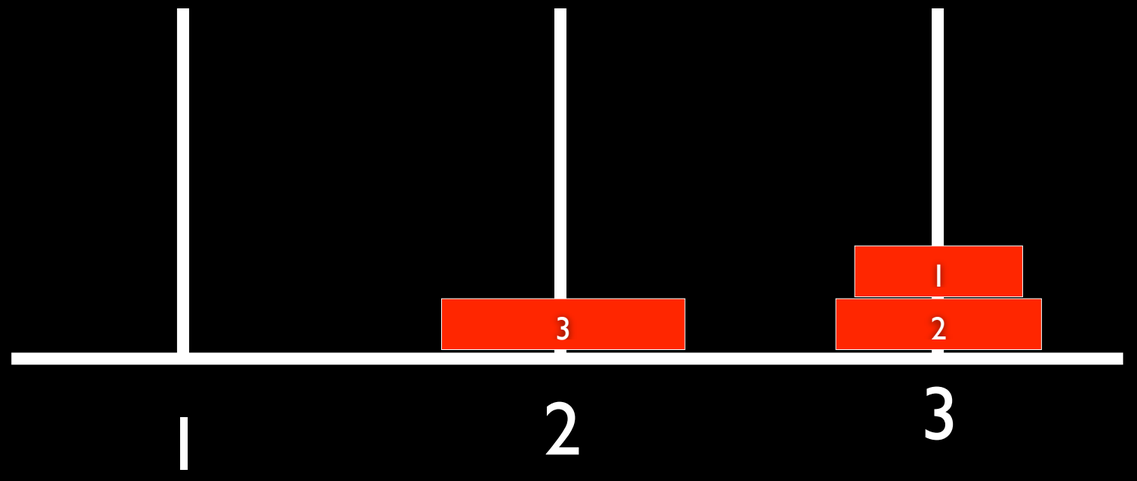
Towers of Hanoi

n = 3: move disks 1&2 from post 1 to 3



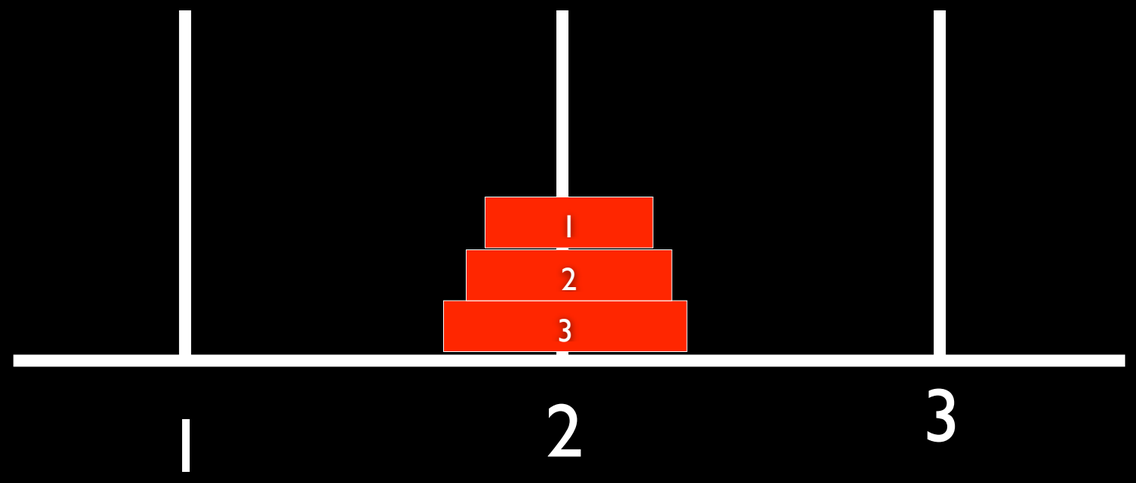
Towers of Hanoi

n = 3: move disks 3 from post 1 to 2

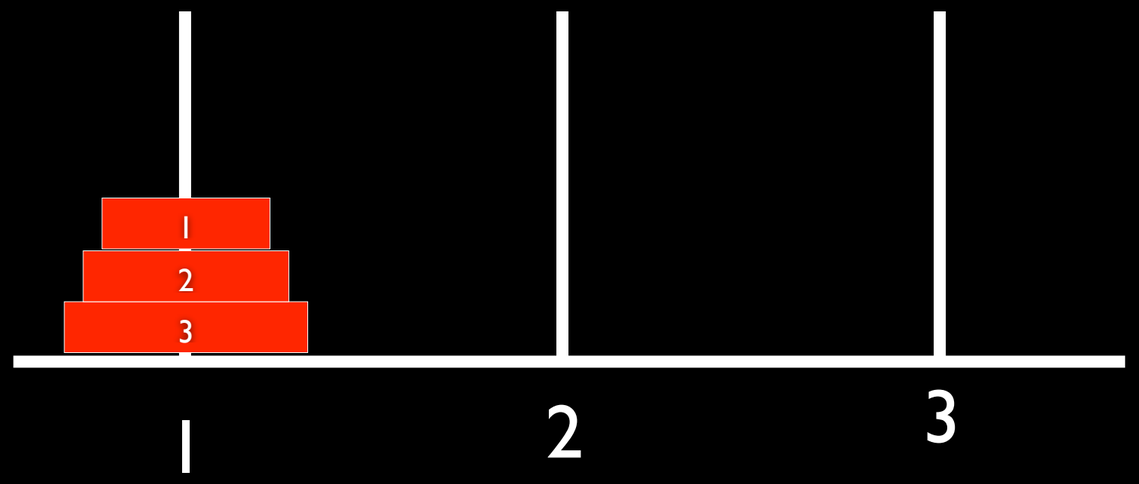


Towers of Hanoi

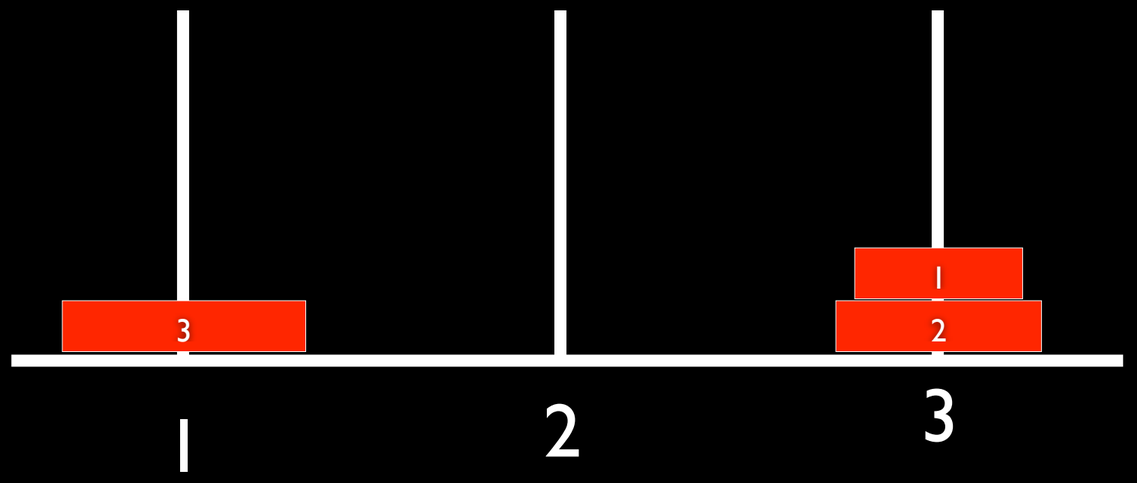
n = 3: move disks 1&2 from post 3 to 2



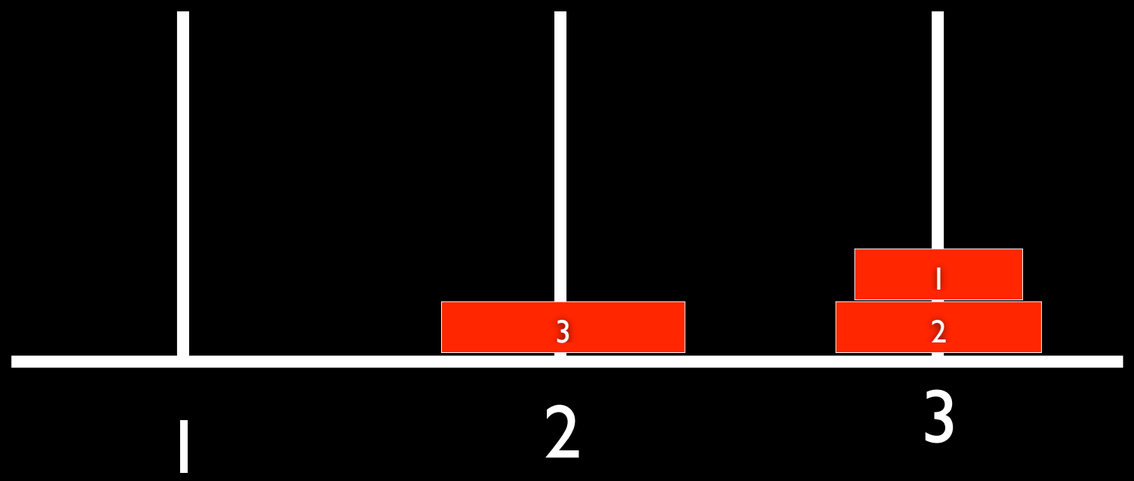
```
hanoi(3,1,2) # move 3 disks from post 1 to 2
```



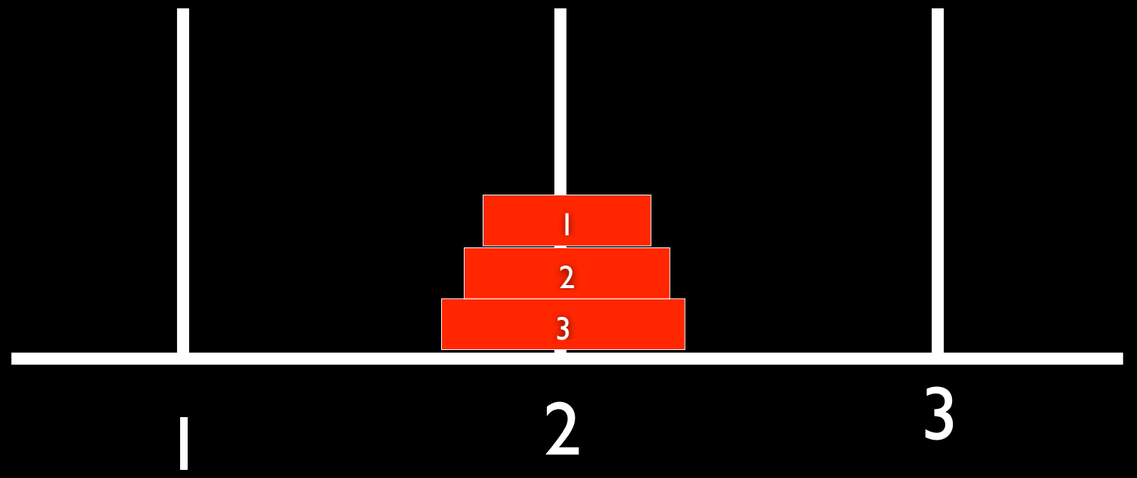
```
hanoi(3,1,2) # move 3 disks from post 1 to 2
  hanoi(2,1,3) # move 2 disks from post 1 to 3
```



```
hanoi(3,1,2) # move 3 disks from post 1 to 2
  hanoi(2,1,3) # move 2 disks from post 1 to 3
  move(3,1,2)  # move disk 3 from post 1 to 2
```



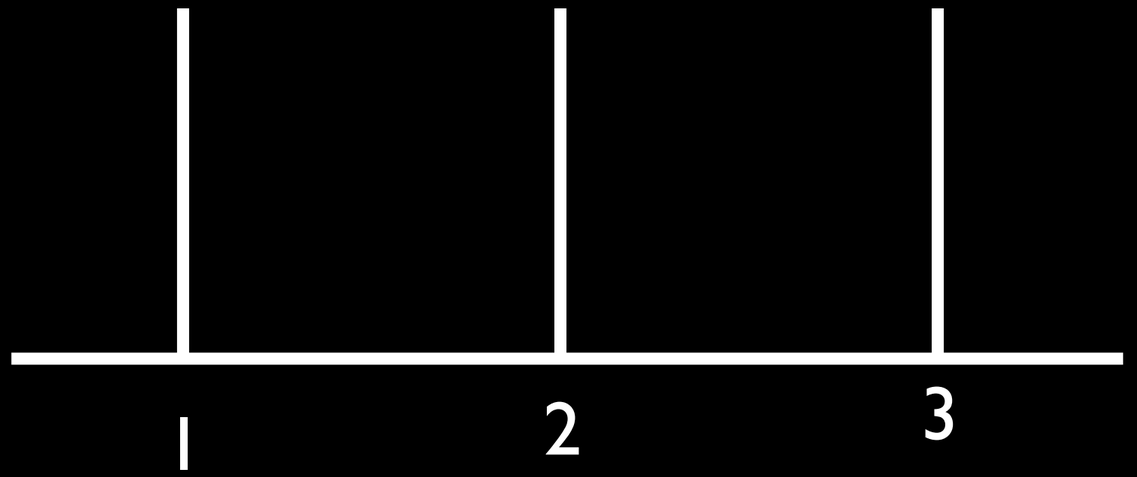
```
hanoi(3,1,2) # move 3 disks from post 1 to 2
  hanoi(2,1,3) # move 2 disks from post 1 to 3
  move(3,1,2)  # move disk 3 from post 1 to 2
  hanoi(2,3,2) # move 2 disks from post 3 to 2
```



```
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
```

```
def move_disk(disk_number, from_peg, to_peg):
    print("Move disk " + str(disk_number) + " from peg " \
        + str(from_peg) + " to peg " + str(to_peg) + ".")


def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
```

```
spare_peg = 6 - start_peg - end_peg
```



```
def move_disk(disk_number, from_peg, to_peg):
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def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
```

```
def move_disk(disk_number, from_peg, to_peg):
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        + str(from_peg) + " to peg " + str(to_peg) + ".")


def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
```

```
def move_disk(disk_number, from_peg, to_peg):
    print("Move disk " + str(disk_number) + " from peg " \
        + str(from_peg) + " to peg " + str(to_peg) + ".")


def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
```

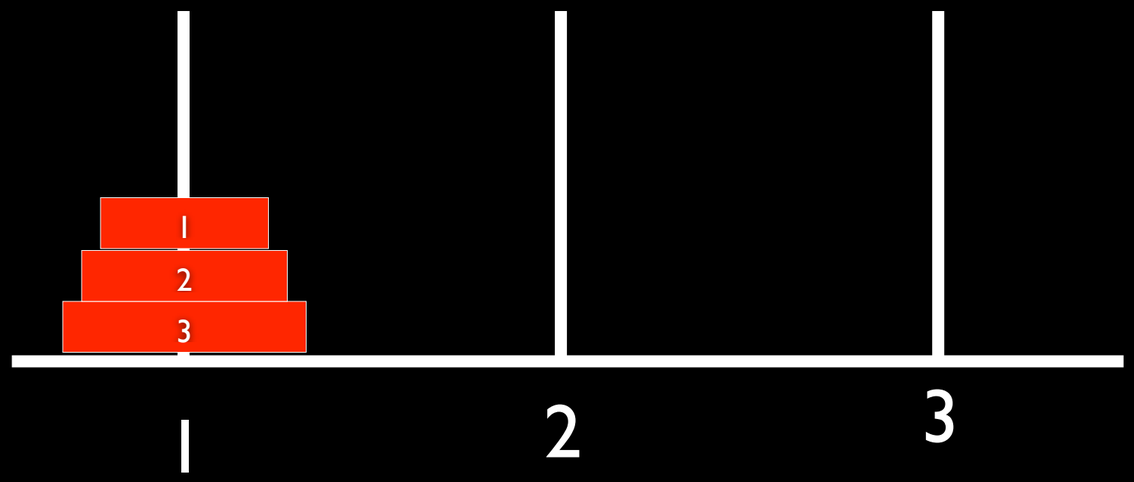
```
def move_disk(disk_number, from_peg, to_peg):
    print("Move disk " + str(disk_number) + " from peg " \
        + str(from_peg) + " to peg " + str(to_peg) + ".")


def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```python
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

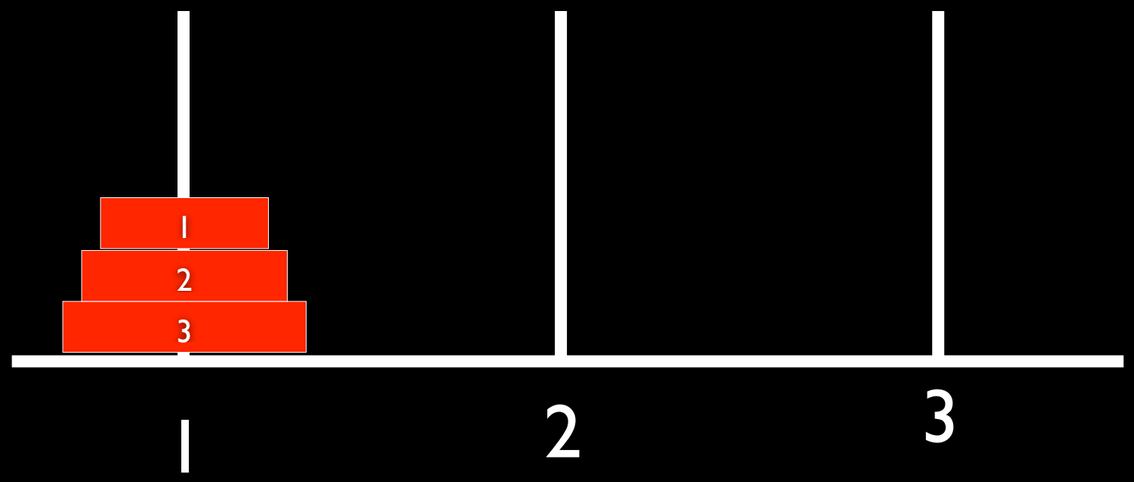
---

hanoi(3,1,2)



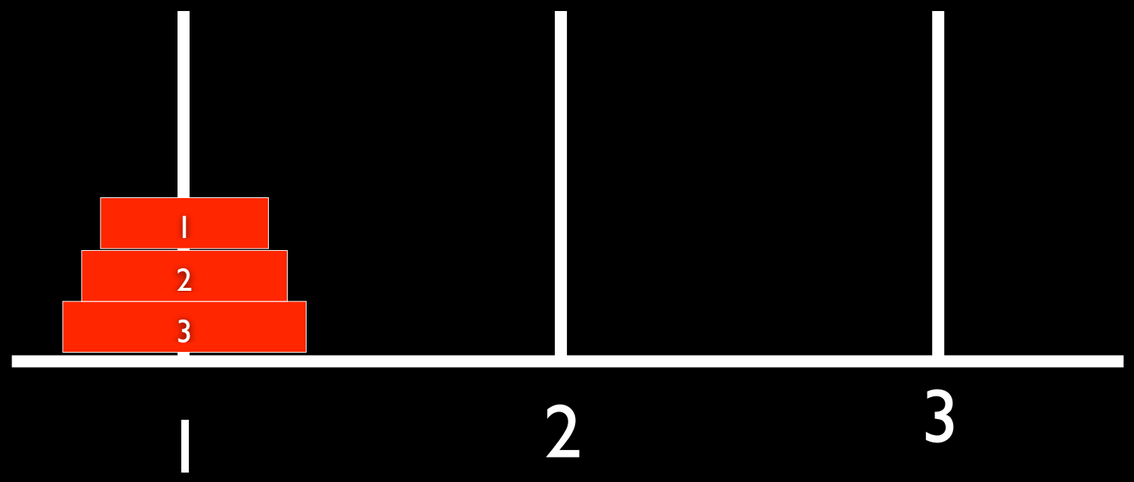
```
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
```



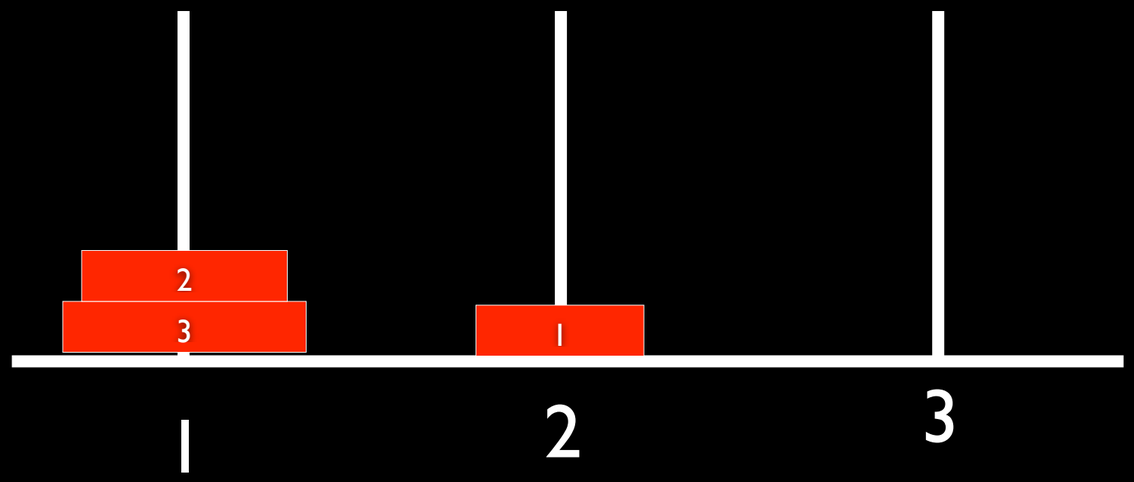
```
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
```



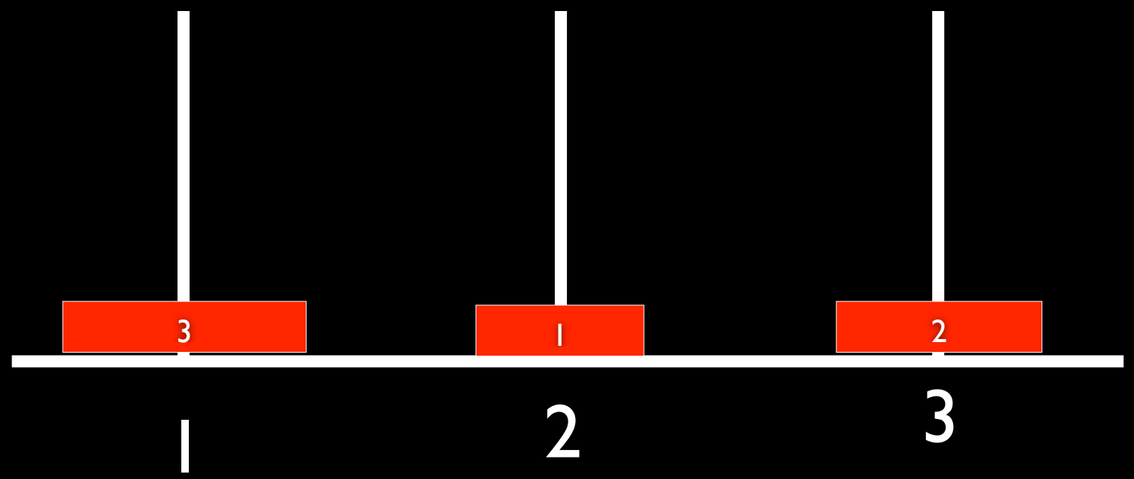
```
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
```



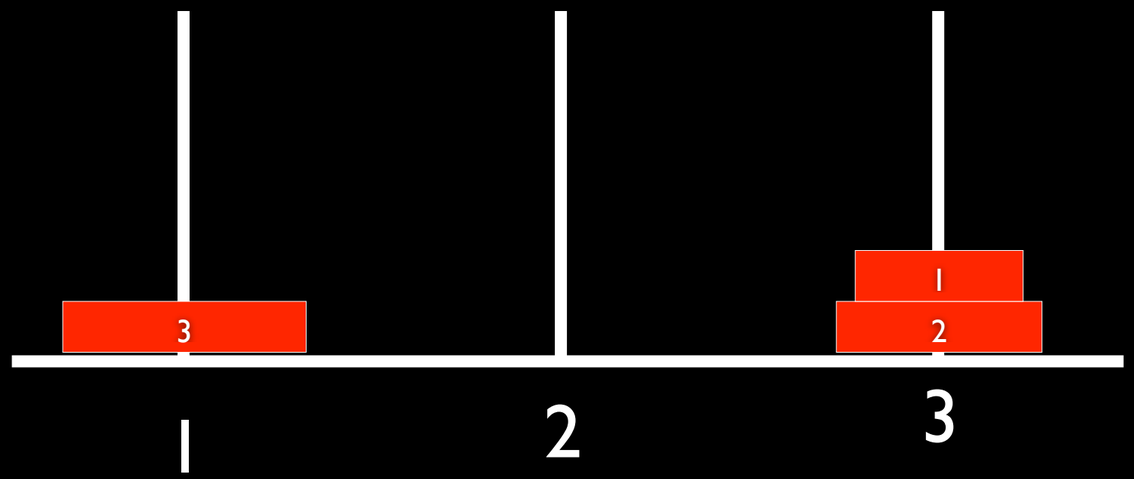
```
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
```



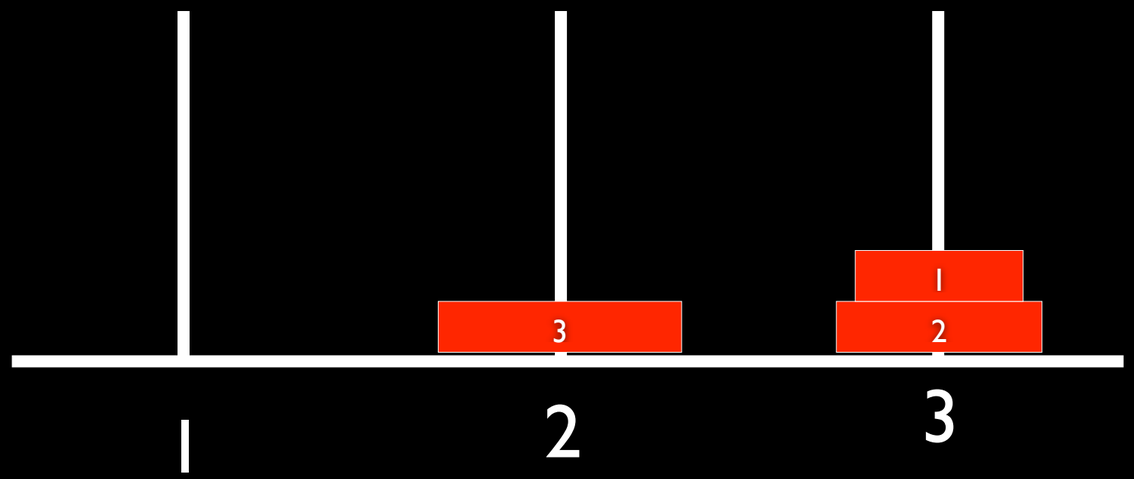
```
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
```



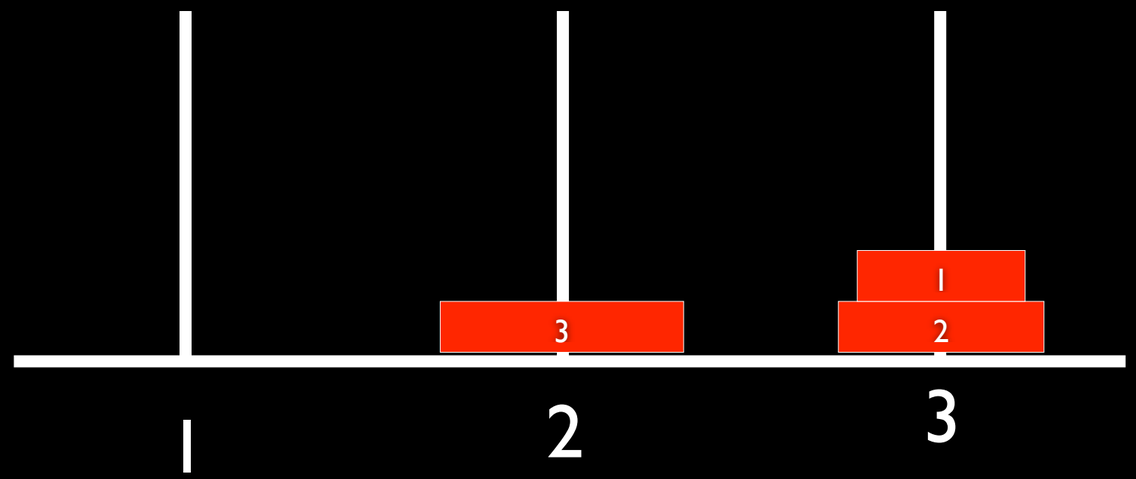
```python
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
```



```python
def solve_hanoi(n, start_peg, end_peg):
    if n == 1:
        move_disk(n, start_peg, end_peg)
    else:
        spare_peg = 6 - start_peg - end_peg
        solve_hanoi(n - 1, start_peg, spare_peg)
        move_disk(n, start_peg, end_peg)
        solve_hanoi(n - 1, spare_peg, end_peg)
```

```
hanoi(3,1,2)
  hanoi(2,1,3)
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
    hanoi(1,3,1)
    move_disk(2,3,2)
    hanoi(1,1,2)
```



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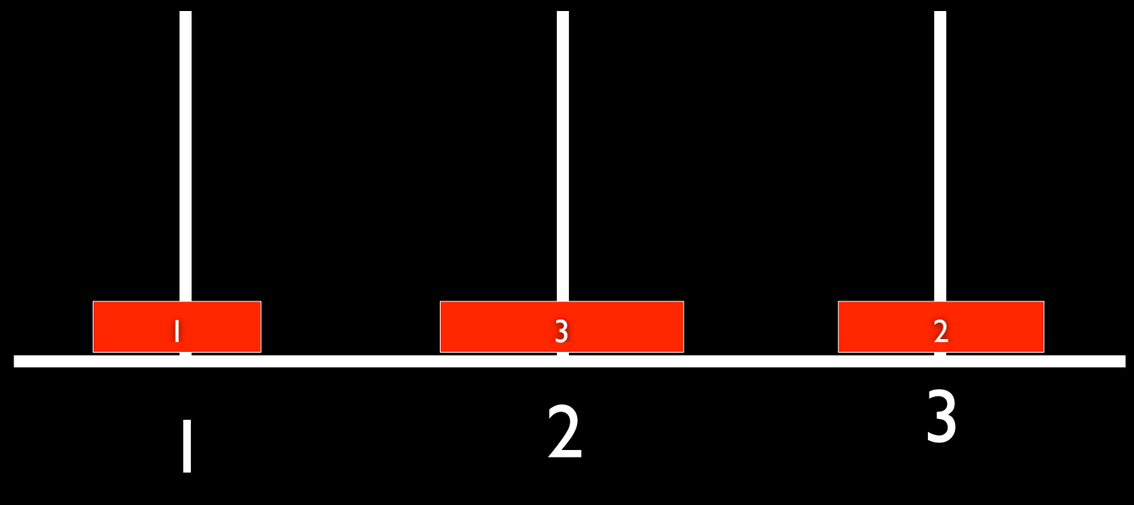
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
    hanoi(1,3,1)
    move_disk(2,3,2)
    hanoi(1,1,2)
```



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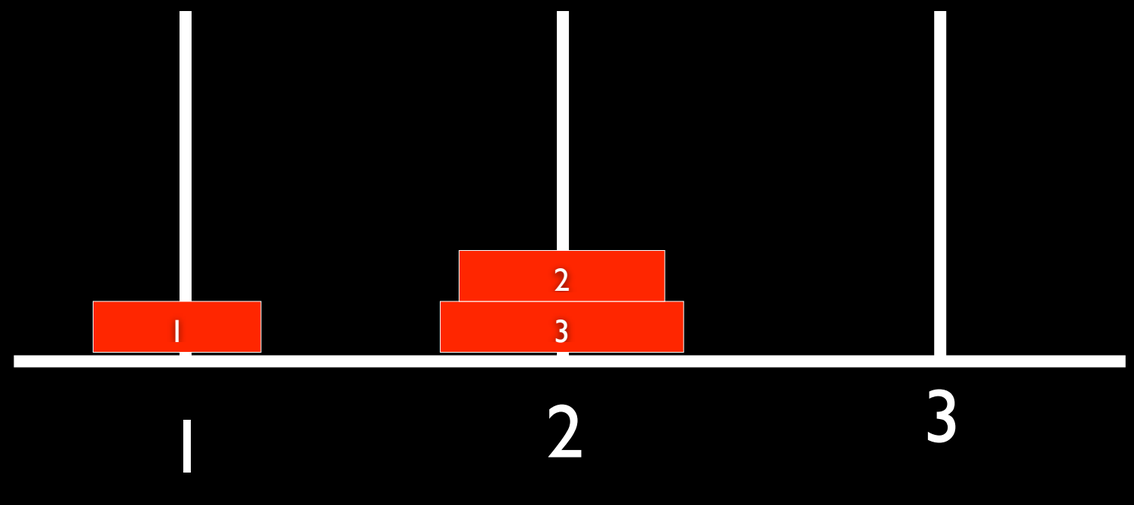
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
    hanoi(1,3,1)
    move_disk(2,3,2)
    hanoi(1,1,2)
```



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```

```
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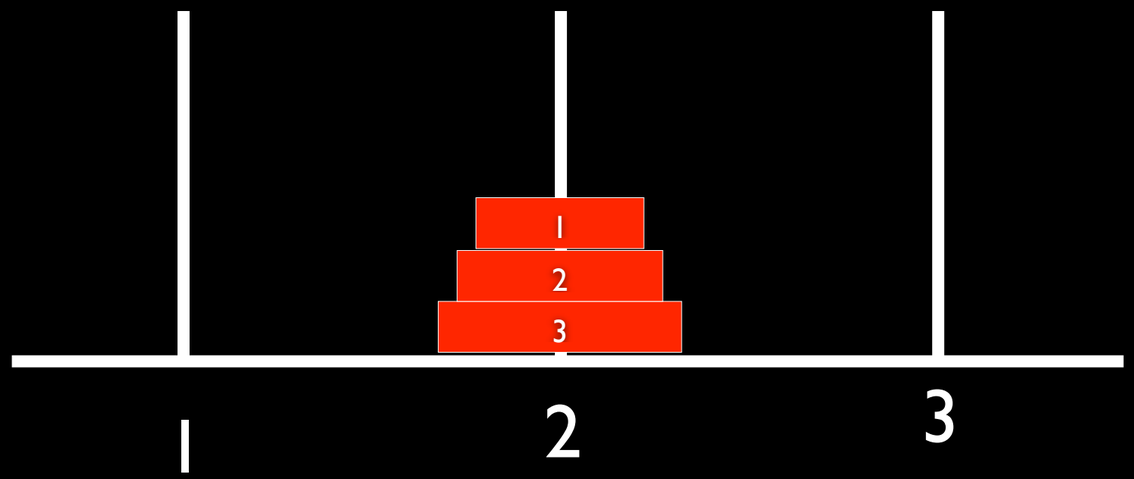
    hanoi(1,1,2)
    move_disk(2,1,3)
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
    hanoi(1,3,1)
    move_disk(2,3,2)
    hanoi(1,1,2)
```



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```
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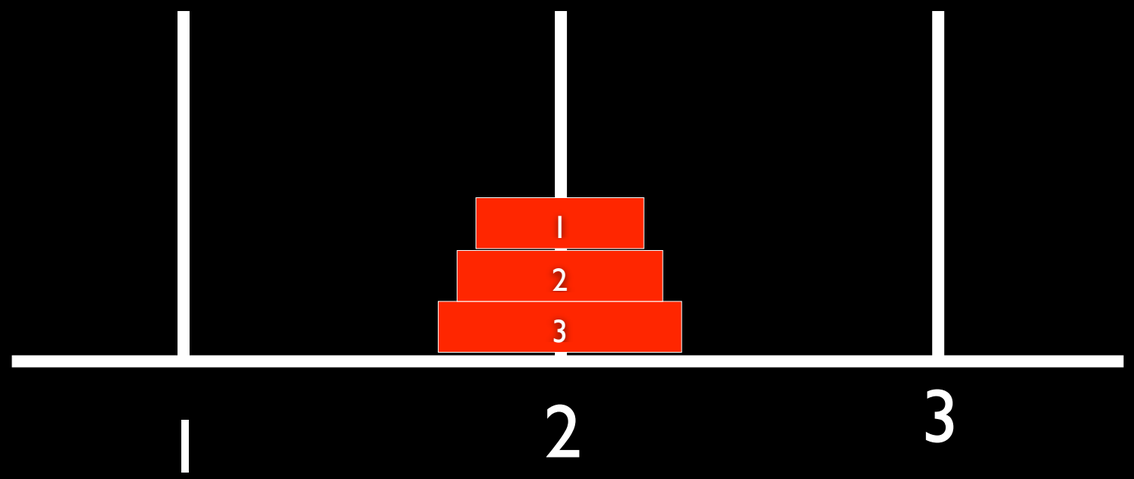
    hanoi(1,2,3)
  move_disk(3,1,2)
  hanoi(2,3,2)
    hanoi(1,3,1)
    move_disk(2,3,2)
    hanoi(1,1,2)
```



discs	moves
1	1
2	3
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
…	
64	

discs	moves
1	1
2	3
3	7
4	
5	
6	
7	
8	
9	
10	
11	
12	
…	
64	

discs	moves
1	1
2	3
3	7
4	15
5	
6	
7	
8	
9	
10	
11	
12	
…	
64	

discs	moves
1	1
2	3
3	7
4	15
5	31
6	63
7	127
8	255
9	511
10	1,023
11	2,047
12	4,095
…	
64	18,446,744,073,709,551,615